Dynamics of voltage-driven Josephson junction arrays

1992 J. Phys.: Condens. Matter 4 4149

(http://iopscience.iop.org/0953-8984/4/16/015)

J. Phys.: Condens. Matter 4 (1992) 4149–4163. Printed in the UK

Dynamics of voltage-driven Josephson junction arrays

Sung-Jong Lee and Thomas C Halsey
The James Franck Institute and Department of Physics, The University of Chicago, 5640 South Ellis Avenue, Chicago, IL 60637, USA

Received 25 November 1991

Abstract. We discuss the dynamics of voltage-driven Josephson junction arrays with negligible capacitances. The array dynamics shows a sensitive dependence on initial phases, leading to current-spike structures in the time-averaged I–V relations. We demonstrate this effect explicitly for the simple system of a two-junction chain, which we also use to illustrate the main features of current response to external driving voltages. We then show that a class of simplified *ansatz* solutions of two-dimensional voltage-driven Josephson junction arrays under a perpendicular magnetic field can be reduced to two-junction dynamics. In both cases, we find that coherent phase-slip processes can lead to subharmonic lockings with an external RF field.

1. Introduction

The dynamics of single Josephson junctions driven by AC currents has provided one of the great laboratories for non-linear dynamics in solid state systems [1–3]. Such dynamical phenomena as period doubling, the quasi-periodic transition to chaos, and hysteresis have been extensively studied with this system. But Josephson junctions can also be used to create spatially extended dynamical systems. Two-dimensional Josephson junction arrays containing up to $\sim 10^6$ individual junctions have been used for many years to study statistical mechanics problems [4–6]. More recently, experimentalists have turned to the remarkable dynamical properties of these arrays [7–9].

The principal result of these studies has been the observation of giant Shapiro steps and fractional giant Shapiro steps in I–V relations of Josephson junction arrays [8]. These giant Shapiro steps and fractional giant Shapiro steps represent a coherent phase-locking of all junctions in an $N \times N$ array at time-averaged voltages satisfying the relation,

$$V_n = n\left(\frac{N\hbar\omega}{2eq}\right) \qquad n \in Z \tag{1.1}$$

with an external magnetic field inducing f $(= p/q;\ p, q$ relatively prime) superconducting flux quanta per unit plaquette and with ω equal to the RF frequency of the driving current. The cases with $n/q \in Z$ represent integer giant Shapiro steps and those with $n/q \notin Z$ represent fractional giant Shapiro steps.

When capacitances are neglected, the current through junction ij in the array is

$$I_{ij} = I_{c,ij}\sin\left(\theta_i - \theta_j - \frac{2e}{\hbar c}\int_i^j \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}\right) + \frac{\hbar}{2eR_{ij}}\left(\dot\theta_i - \dot\theta_j - \frac{2e}{\hbar c}\int_i^j \dot{\boldsymbol{A}}\cdot \mathrm{d}\boldsymbol{r}\right) \tag{1.2}$$

0953-8984/92/164149+15$04.50

where θ_i, θ_j are the superconducting phases of islands i and j respectively and A is the vector potential. R_{ij} denotes the normal state resistance of the junction, and $I_{c,ij}$ its critical current. Equation (1.2) combined with Kirchhoff's laws completely determines the dynamics of the array.

Numerical simulations based on this coupled RSJ model have reproduced the previously mentioned step structures [10]. However, analytic approaches to the phase dynamics of the arrays, especially in the experimentally relevant current-driven regime, have been scarce. Voltage-driven arrays under magnetic fields were analysed by Halsey using a special *ansatz* for the ground state configuration, that of 'staircase states' (see section 3) [11, 12]. By using an adiabaticity assumption, not only fractional giant Shapiro steps but also additional subharmonic giant Shapiro steps were predicted with estimates of stepwidths in some limiting situations. These subharmonic giant Shapiro steps would occur at voltages satisfying

$$V_{n,m} = \frac{n}{m}\frac{N\hbar\omega}{2eq} \qquad n, m \in Z. \tag{1.3}$$

These predictions have not fared particularly well by comparison with experimental results. In particular, subharmonic steps have been elusive experimentally, and it is possible that their appearance is related to self-field effects not included in the RSJ model of equation (1.2) [13]. The theory makes specific predictions only for currents oriented along a diagonal axis of a square lattice; experimental studies in this geometry have not only seen no subharmonic steps, but no fractional steps either [14, 15]. It is thus profitable to re-examine in detail the specific assumptions of Halsey's work, in order to explore possible sources of the disagreement with experiment.

Halsey's approach is based on two physical assumptions. The first is that a voltage-driven array (far more convenient theoretically) can be substituted for the current-driven experimental situation. The second assumption is that, since the coupled RSJ equations generate a dissipative dynamical system, in the limit where the array dynamics is arbitrarily fast compared with changes in boundary conditions, the state can be modelled as an adiabatically changing metastable state. As discussed in [11], this implies the existence of rapid phase-slip processes when the adiabatically changing metastable state enters an unstable region of phase space.

In this study we shall explore the validity of these assumptions by studying some examples of voltage-driven systems in which the dynamics can be partially or completely solved. Our principal result is that the dynamics of voltage-driven arrays is considerably more subtle than the dynamics of current-driven arrays, even in the limiting case of a two-junction-in-series array. The dynamics of voltage-driven arrays is far more dependent on initial conditions, which can alter the current even well away from any rational voltage step. Nevertheless, dependence of the current on the phase of the external driving voltage appears only for subharmonic or harmonic voltages satisfying equation (1.3). In numerical simulations of the voltage-driven case, current-spike structures appear that do seem to have at least a qualitative relationship to the current-driven Shapiro steps.

This can be understood by reviewing the case of the single junction. A single current-driven junction obeying equation (1.2) under a sinusoidal RF field has a well defined voltage at every value of the current. The Shapiro steps appear only for harmonic voltages,

$$V_n = \frac{\hbar\omega}{2e} n. \tag{1.4}$$

If the junction is voltage-driven, for voltages not satisfying equation (1.4), there is no net current flow in the supercurrent channel, i.e. $\langle \sin\theta_{ij}\rangle = 0$, where the averaging is over time. However, as originally realized by Josephson [16], the situation is different for voltages satisfying equation (1.4). In these cases, $\langle \sin\theta_{ij}\rangle \neq 0$ necessarily, rather the current in the supercurrent channel is determined by the phase shift between the RF driving field and the Josephson oscillations of the junction.

This suggests that if we wish to generalize voltage-driven results to the current-driven situation, one possible procedure is to look for values of the voltage where variable currents are possible, depending upon the phase relationships of the system. For systems of more than one junction, however, this is an ambiguous procedure. Not only is there an intrinsic phase of the external driving voltage, there are also intrinsic phases of the initial state of the junction network or array. As we shall see later, these intrinsic phases can alter the average current, even at voltages completely unrelated to the RF driving frequency.

In this case, it is natural to hypothesize that it is dependence on the phase of the driving field that will translate into a phase-locking in the current-driven situation. We shall find later that this implies both harmonic and subharmonic lockings in even a two-junction network, provided that phase-slip processes can relax sufficiently quickly.

The relaxation time of the phase-slip processes scales as $(2/V_j)^{1/2}$ (V_j is the average DC voltage per junction) which becomes much less than the Josephson oscillation time ($= 1/V_j$) in the limit where V_j is small. Even outside this adiabatic limit, there is still a dependence of the current on the RF driving phase; however, the size of the predicted step structures decreases as one leaves the adiabatic regime. This suggests that the non-appearance of subharmonic lockings in experiment is due either to a quantitative problem, i.e. the distance of the experimental situation from the adiabatic limit, or to some, as yet poorly understood, suppression of the subharmonic lockings due to the current driving.

This paper contains four sections and an appendix. In section 2 we analyse the voltage-driven two-junction system with equal resistances and critical currents. This system admits a closed-form solution involving integrals of transcendental functions, allowing analysis of some of its dynamical features. In section 3 staircase-*ansatz* dynamics is used to investigate phase-slip processes in a two-dimensional array in a magnetic field. In section 4 we discuss the relation of these results both to Halsey's original conclusions and to the experimentally relevant current-driven situation. In the appendix we discuss useful approximations to expressions for the current appearing earlier in the paper.

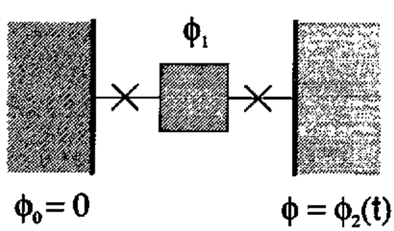

Figure 1. A system of two Josephson junctions in series. The phases ϕ_0 and ϕ_2 in the boundary regions are imagined to be fixed. The phase on the centre island is ϕ_1.

2. Voltage-driven two-junction system

The simplest non-trivial voltage-driven Josephson junction system consists of two junctions coupled in series, with the boundary phases ϕ_0 and ϕ_2 given functions of time (see figure 1). Using current conservation through the island in the middle, and assuming homogeneity of I_c and R, we get

$$\frac{\hbar}{2eR}(\dot{\phi}_1 - \dot{\phi}_0) + I_c \sin(\phi_1 - \phi_0) = \frac{\hbar}{2eR}(\dot{\phi}_2 - \dot{\phi}_1) + I_c \sin(\phi_2 - \phi_1). \tag{2.1}$$

We choose $\phi_0 = 0$; the voltage is given in terms of $\phi_2(t)$ by $\hbar\dot{\phi}_2/2e = V(t)$. We are here making the unrealistic assumption that the source impedance $R_{ext} = 0$.

We use reduced units of time and current respectively: $\bar{t} = 2eI_c Rt/\hbar$, $i(\bar{t}) = I(\bar{t})/I_c$, and $v(\bar{t}) = V(\bar{t})/I_c R$. Later, we will introduce the scaled RF frequency $\bar{\omega}$, which will be related to the physical frequency ω by $\bar{\omega} = \hbar\omega/2eI_c R$. Equation (2.1) now simplifies to

$$\dot{\phi}_1 + \sin(\phi_1) = v(\bar{t}) - \dot{\phi}_1 + \sin(\phi_2 - \phi_1) \tag{2.2}$$

where the dot now refers to a derivative with respect to the scaled time $\bar{t}$. Now define $y \equiv (\phi_2 - 2\phi_1)/2$. The case $y = 0$ corresponds to equal phase differences across each of the two junctions. After some rearrangements, we obtain

$$\dot{y} = -\cos\left(\frac{\phi_2}{2}\right)\sin(y). \tag{2.3}$$

This can be integrated to give

$$\int_{y_0}^{y} \frac{\mathrm{d}y}{\sin(y)} = -\int_{\bar{t}_0}^{\bar{t}} \cos\left(\frac{\phi_2}{2}\right) \mathrm{d}\bar{t}. \tag{2.4}$$

Thus by choosing $\bar{t}_0 = 0$ and the branch $-\pi \leqslant y < \pi$, we have

$$y(\bar{t}) = 2\arctan\left\{c_0 \exp\left[-\int_0^{\bar{t}} \cos\left(\frac{\phi_2}{2}\right) \mathrm{d}\bar{t}\right]\right\} \tag{2.5}$$

with

$$\begin{aligned} \phi_2 &= \int_0^{\bar{t}} v(\bar{t})\,\mathrm{d}\bar{t} + \beta_0 \\ c_0 &= \tan\left(\frac{y_0}{2}\right) \\ y_0 &= y(\bar{t} = 0). \end{aligned} \tag{2.6}$$

From this result and the relation $y = (\phi_2 - 2\phi_1)/2$, we can obtain an expression for ϕ_1.

The current as a function of time is

$$i(\bar{t}) = \sin(\phi_1) + \dot{\phi}_1 = \sin\left(\frac{\phi_2}{2} - y(\bar{t})\right) + \frac{\dot{\phi}_2}{2} - \dot{y}(\bar{t}). \tag{2.7}$$

By using trigonometric identities and the previous expression for $y(\bar{t})$ in equation (2.5) we arrive at the following expression.

$$i(\bar{t}) = \frac{\dot{\phi}_2}{2} + \left(\sin\frac{\phi_2}{2}\right)\frac{1 - c_0^2 \exp[-2G(\bar{t})]}{1 + c_0^2 \exp[-2G(\bar{t})]} \tag{2.8a}$$

where

$$G(\bar{t}) \equiv \int_0^{\bar{t}} \cos(\phi_2/2)\, \mathrm{d}\bar{t} \tag{2.8b}$$

and c_0 is defined above.

To begin with, let us first consider some general features of $y(\bar{t})$ and $i(\bar{t})$. From equations (2.5) and (2.6), which can be put in the form,

$$\tan\left(\frac{y}{2}\right) = \tan\left(\frac{y_0}{2}\right)\exp[-G(\bar{t})] \tag{2.9}$$

we see that if $y_0 = 0$, then $y(\bar{t}) = 0$ for all $\bar{t}$. Furthermore, since $\exp[-G(\bar{t})] \geqslant 0$, $\tan(y/2)$ (and hence y, too) does not change its sign. In other words, if its initial value is > 0 (< 0) then it remains > 0 (< 0) for all later times. This holds for all possible forms of $v(\bar{t})$.

Until now, our arguments did not depend on the time dependence of the driving voltage. Now we restrict our attention to the case of DC plus AC voltage driving with frequency $\bar{\omega}$,

$$v(\bar{t}) = 2[v_0 + v_1 \sin(\bar{\omega}\bar{t})]. \tag{2.10}$$

Then $G(\bar{t})$ becomes

$$\begin{aligned} G(\bar{t}) &= \int_0^{\bar{t}} \cos\left(\frac{\phi_2}{2}\right)\mathrm{d}\bar{t} = \int_0^{\bar{t}} \cos\left[\beta_0 + v_0\bar{t} - \frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t})\right]\mathrm{d}\bar{t} \\ &= \mathrm{Re}[\exp(\mathrm{i}\beta_0)\sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m\left(-\frac{v_1}{\bar{\omega}}\right)\int_0^{\bar{t}} \exp[\mathrm{i}(v_0 - m\bar{\omega})\bar{t}]\,\mathrm{d}\bar{t}]. \end{aligned} \tag{2.11}$$

Note that β_0 fixes the phase relation between the RF driving and the Josephson oscillations. If $v_0 \neq m\bar{\omega}$ then

$$G(\bar{t}) = \sum_{m=-\infty}^{\infty} \frac{J_m(-v_1/\bar{\omega})}{(v_0 - m\bar{\omega})}\left\{\sin\left[(v_0 - m\bar{\omega})\bar{t} + \beta_0 + \frac{m\pi}{2}\right] - \sin\left[\beta_0 + \frac{m\pi}{2}\right]\right\}. \tag{2.12}$$

This is a superposition of oscillating functions with frequencies,

$$\nu_m = v_0 - m\bar{\omega} \qquad m \in Z. \tag{2.13}$$

$G(\bar{t})$ will be bounded between $G_{\max}$ and $G_{\min}$, as it is a sum of oscillating terms. Therefore $\tan(y/2)$ and y also oscillate between two finite values. On the other hand, if $v_0 = m\bar{\omega}$ for some integer m, then in equation (2.11) for $G(\bar{t})$, one term (the mth term) is replaced by a linear term in $\bar{t}$ with a positive or a negative coefficient depending on m, β_0, $\bar{\omega}$, v_0 and v_1. As $\bar{t}$ gets larger, this linear term will dominate $G(\bar{t})$. Therefore, if $G(\bar{t})$ is linearly increasing, then y goes to $\pm\pi$ as $\bar{t} \to \infty$, and if $G(\bar{t})$ is linearly decreasing, then y goes to zero as $\bar{t} \to \infty$.

Let us consider the behaviour of $i(\bar{t})$, especially the frequency spectrum of $i(\bar{t})$ and time-averaged currents. We begin with the case when $y_0 = 0 = c_0$, i.e. when

$$\phi_2 - \phi_1 = \phi_1 - \phi_0. \tag{2.14}$$

This corresponds to the case where the initial phase differences of the two junctions are equal to each other and later developments of the phase differences of each junction are also equal to each other, giving the same dynamics as that of a single junction voltage-driven system. This dynamics is closer to the current-driven dynamics than is the case for other values of y_0. For this case, from equation (2.8),

$$i(\bar{t}) = \frac{\dot{\phi}_2}{2} + \sin\left(\frac{\phi_2}{2}\right) \qquad \dot{\phi}_2 = 2[v_0 + v_1 \sin(\bar{\omega}\bar{t})]. \tag{2.15}$$

This is a superposition of AC currents with frequencies

$$\nu_m = v_0 - m\bar{\omega} \tag{2.16}$$

where m is an integer, with an additional DC current in the normal current channel. Only when $v_0 = m\bar{\omega}$ does the time-averaged current have a finite range of different values depending on β_0. (This situation is the same as for a single junction under voltage driving.)

When $c_0 \neq 0$, $i(\bar{t})$ has a much more complicated behaviour due to the term $\exp[-2G(\bar{t})]$. Using the results for $G(\bar{t})$ in previous pages, we can put

$$-2G(\bar{t}) = D + \sum_{m=-\infty}^{\infty} F_m \sin\left[(v_0 - m\bar{\omega})\bar{t} + \beta_0 + \frac{m\pi}{2}\right] \tag{2.17}$$

where

$$\begin{aligned} D &\equiv 2 \sum_{m=-\infty}^{\infty} \frac{J_m(-v_1/\bar{\omega})}{v_0 - m\bar{\omega}} \sin\left(\beta_0 + \frac{m\pi}{2}\right) \\ F_m &\equiv -2\frac{J_m(-v_1/\bar{\omega})}{v_0 - m\bar{\omega}}. \end{aligned} \tag{2.18}$$

If we examine equation (2.8) for the current $i(\bar{t})$, we see that the supercurrent channel involves two factors.

(i) The term $\sin(\phi_2(\bar{t}))$. This term will contain frequency modes $\nu_1 = v_0 - m_1\bar{\omega}$, where $m_1 \in Z$.

(ii) The term involving $G(\bar{t})$. From equations (2.17) and (2.18), we see that this term will contain frequency modes $\nu_2 = l_2(v_0 - m_2\bar{\omega})$, where $l_2, m_2 \in Z$.

Hence the frequency modes for $i(\bar{t})$ are given by all possible combinations of the Josephson frequency and the driving frequency,

$$\nu_{n_1,n_2} = n_1 v_0 - n_2\bar{\omega} \qquad n_1, n_2 \in Z \tag{2.19}$$

in contrast to the single junction case, in which the Josephson frequency is combined only with harmonics of the driving frequency.

In general, the time-averaged DC current due to the supercurrent channel is non-vanishing; for both subharmonic and harmonic values of the driving voltage, its value depends upon the initial phase β_0. This can be seen by examining a typical term in the expansion of equation (2.8*a*) in a product of Bessel series. We shall see this for an explicit example in the appendix. Note, however, that D and F_m become small in the limit of large v_0, which is the opposite of the adiabatic case discussed in [11]. In this limit, $G(\bar{t}) \to 0$, and equation (2.8*a*) reduces to the single-junction form, albeit with changed parameters. Thus in this case there will be no subharmonic lockings.

If v_0 and $\bar{\omega}$ are irrationally related then we can see from the previous argument that the frequency modes are a dense subset of the real numbers. The time-averaged current (in this case due to the superconducting channel) is also in general non-vanishing, though a quantitative estimate of this current is rather difficult. It is, however, easy to see that although it depends upon the initial value of y, it does not depend upon the phase β_0 of the AC driving. Again, an explicit example of this is discussed in the appendix.

Finally, from equations (2.9) and (2.11) we can estimate the time for y to change significantly from its initial value y_0 in a phase-slip process. Suppose $\phi_2(\bar{t}) = \pi$ at $\bar{t} = 0$, and that it is increasing at this time. Then if $v_1 \ll v_0$, we have approximately

$$\cos\left(\frac{\phi_2(\bar{t})}{2}\right) = -\sin\left(v_0\bar{t} - \frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t})\right) \approx -v_0\bar{t} + \frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t}) \approx -v_0\bar{t} \qquad (2.20)$$

so that

$$G(\bar{t}) \approx -v_0\bar{t}^2/2 \qquad (2.21)$$

for small times. Thus the time $\bar{t}_1$ at which $y(\bar{t})$ will have changed from y_0 to y_1 is determined by

$$\bar{t}_1 \approx \sqrt{\frac{2}{v_0}}\sqrt{\log\left[\tan\left(\frac{y_1}{2}\right) \Big/ \tan\left(\frac{y_0}{2}\right)\right]} \qquad (2.22)$$

so that the phase-slip time $\bar{\tau}_S \approx \sqrt{2/v_0}$. This may be compared with an intrinsic relaxation time $\bar{\tau}_R$, which is equal to one in reduced units; ($\tau_R = \hbar/2eI_c R$ in physical units). When $\bar{\tau}_S \ll 2/v_0$, the time scale for ϕ_2 to change, then we are in the adiabatic limit.

In the appendix, we use an approximation to estimate the time-averaged currents. Numerically, we can evaluate the range of time-averaged currents for various external driving voltages. This can be done by directly integrating the equations of motion, using a fixed time-varying external voltage $V(t)$. We used a first-order time integration of the equation of motion (2.3) (which gives more information about the current than higher-order methods). For a given external voltage, maximum and minimum currents were obtained by varying the initial phase configuration (β_0, y_0). We generally took about 200–400 such initial points, and then took the maximum and minimum currents over this set of initial configurations. Figure 2 shows maximum and minimum currents in terms of the DC component of external voltage. We find a non-vanishing range of time-averaged current not only for integer harmonic voltages

$V_{j;n}$ per junction, $V_{j;n} = n\hbar\omega/2e$, but also for subharmonic voltages, as well as for voltages with no obvious relation to the driving frequency. We see that, at low-order fractional voltages such as $V_j = \frac{1}{2}(\hbar\omega/2e)$ or $\frac{1}{3}(\hbar\omega/2e)$, there are indications of strong variation of the range of currents with voltage. It is not clear whether the current range against time-averaged voltage varies smoothly or not with voltage; it may be discontinuous at rational values of DC voltage.

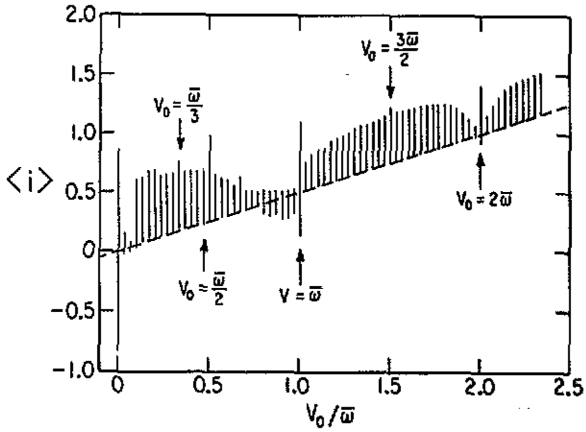

Figure 2. The maximum and minimum time-averaged currents plotted against DC voltage in a two-junction system under voltage-driving conditions. The broken line shows the current in the normal channel; the full lines show the range of possible average currents. Structure is visible at both harmonic and subharmonic voltages. These results were obtained using a first-order integration scheme, with a time step of $0.005 \times \tilde{\tau}_R$, and an integration time of 1000–$2000 \times \tilde{\tau}_R$. The value of v_1 was taken to be $v_1 = 0.4$, and $\bar{\omega}$ was taken to be $\bar{\omega} = 0.5$.

3. Dynamics of two-dimensional diagonal arrays

The ground-state configurations of two-dimensional square Josephson junction arrays in a magnetic field are not, in general, known except for a few relatively simple values of flux per plaquette $f = p/q$. For some of these values of f, the ground states are so-called staircase states, which are much simpler to deal with analytically due to a symmetry of the vortex and current configurations. We begin with a brief review of these states.

In order to find the ground-state configuration of a Josephson junction array in an external magnetic field with $f = p/q$, we have to minimize the Hamiltonian

$$H = -\sum_{\langle ij\rangle} \frac{\hbar I_{c,ij}}{2e} \cos(\theta_i - \theta_j - A_{ij}) \tag{3.1}$$

in terms of the phases $\{\theta_i\}$. Here, the sum is over nearest neighbours. A_{ij} is the line integral of the vector potential, as in equation (1.2),

$$A_{ij} = \frac{2e}{\hbar c} \int_i^j \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r}. \tag{3.2}$$

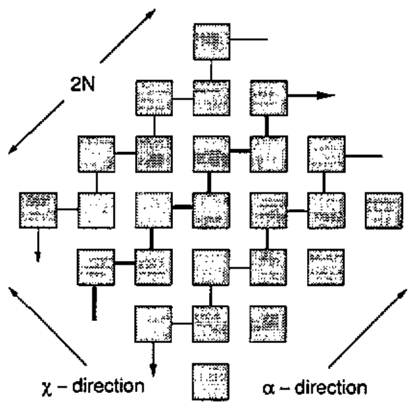

Figure 3. A square array showing the diagonal staircases along which the current is constant (zig-zag lines), as well as the α- and χ-directions. The sample is imagined to be $2N \times 2N$, so that the number of junctions in a staircase is $2N$.

This satisfies $\sum_P A_{ij} = 2\pi f$ with the sum in the counter-clockwise direction around a plaquette.

We deal with uniform arrays only with $I_{c,ij} = I_c$. The extremum condition for H is equivalent to the requirement that supercurrent be conserved at every site in the array. The supercurrent across the $\langle ij \rangle$ bond is $I_c \sin(\theta_i - \theta_j - A_{ij})$. A simple way to satisfy current conservation in a square array is to require that the supercurrent along any individual diagonal 'staircase' of the array be constant (see figure 3). In this case all of the junctions along a staircase have the same gauge-invariant phase differences. Let us denote these phase differences for the mth staircase as $\phi_m \equiv \theta_i - \theta_j - A_{ij}$. Then we can find locally stable states with

$$\phi_m = \pi f m + \alpha_0 - \pi[fm + \alpha_0/\pi]_n \tag{3.3}$$

where $[x]_n \equiv \mathrm{int}[x + \frac{1}{2}]$ is the nearest integer function. Here, α_0 is determined by minimizing the global energy or, equivalently, by letting the net current be equal to zero. In this way, we get $\alpha_0 = 0$ for odd q, and $\alpha_0 = \pi/2q$ for even q. See [12] for details. For general values of α_0, we obtain staircase states with non-zero net current along the direction of the staircases. We will call this the α-direction, the direction of current obtained by varying α.

Let us assume that an AC plus DC voltage $v(\bar{t}) = 2N(v_0 + v_1 \sin(\bar{\omega}\bar{t}))$ is applied along the α-direction of a diamond-shaped $2N$ by $2N$ array where $2N$ refers to the number of junctions along longitudinal and transverse directions; we suppose that N is an integral multiple of q. Here, again as in the two-junction system in section 2, we use reduced units for notational convenience. Note, in particular, that frequency is measured in units of the intrinsic response frequency $\omega_R \equiv 1/\tau_R \equiv 2eI_cR/\hbar$ and time in units of τ_R. We can construct a simple solution for the dynamics of the array driven by this voltage by adding a time-dependent function to equation (3.3). That is

$$\phi_m(\bar{t}) = \pi f m + \alpha_0 - \pi[fm + \alpha_0/\pi]_n + \alpha(\bar{t}) \tag{3.4}$$

with $\dot{\alpha}(\bar{t}) = v_0 + v_1 \sin(\bar{\omega}\bar{t})$. This can be integrated as

$$\alpha(\bar{t}) = a_0 + v_0\bar{t} - \frac{v_1}{\bar{\omega}} \cos(\bar{\omega}\bar{t}) \tag{3.5}$$

where a_0 is an integration constant.

In this solution, the phase dynamics is such that superconducting currents and normal currents are separately conserved at each node. This is an analogue of the uniform phase dynamics ($y_0 = 0$) of the two-junction system in section 2, extended to two-dimensional arrays. The phase-slip *ansatz* of Halsey is based on the instability of this solution. When $\alpha(\bar{t})$ reaches a value such that, for some m, $\phi_m = \pm\pi/2$, the global energy of the system will be reduced by a phase-slip of π uniformly across the mth staircase, with a corresponding vortex motion across that staircase. Halsey [11] assumed that such a phase-slip took place instantaneously, and proceeded to calculate properties of the Shapiro steps given such an assumption.

However, within the framework of the staircase *ansatz* alone, equation (3.4) is the only solution for $\alpha(\bar{t})$; therefore, using only the staircase *ansatz*, phase-slip processes cannot be analysed in any detail. This is because, in order to see the instability, we must compare the energy of a staircase state with that of a neighbouring state generated through small distortions of phase configurations. These distortions lie outside of those states described by $\alpha(\bar{t})$ alone.

We can generalize this *ansatz* to states with non-zero net current along the direction perpendicular to the α-direction by twisting the phases on successive diagonal planes by constant angles χ_m, $m = 1, \ldots, q$ such that the differences of phase shifts for neighbouring diagonal staircases are

$$\gamma_m \boxminus \chi_m - \chi_{m-1}. \tag{3.6}$$

The direction of current induced by this twist we call the χ-direction; it is perpendicular to the α-direction. These states no longer have uniform current (or uniform gauge-invariant phase difference) along a staircase but rather, horizontal bonds and vertical bonds along a staircase have different, but still constant currents. We will call these generalized staircase states since vorticity along a staircase is constant. By giving a time dependence to both $\gamma_m(\bar{t})$ and $\phi_m(\bar{t})$, we can represent a dynamic state that preserves the generalized staircase form. Using current conservation in the χ-direction, we obtain in reduced units

$$\begin{aligned} -\sin(\phi_m+\gamma_m)+\sin(\phi_m-\gamma_m)-(\dot{\phi}_m+\dot{\gamma}_m)+(\dot{\phi}_m-\dot{\gamma}_m) &= -\sin(\phi_{m+1}+\gamma_{m+1}) \\ &+ \sin(\phi_{m+1}-\gamma_{m+1})-(\dot{\phi}_{m+1}+\dot{\gamma}_{m+1})+(\dot{\phi}_{m+1}-\dot{\gamma}_{m+1}) \end{aligned} \tag{3.7}$$

which can be simplified to

$$\cos(\phi_m)\sin\gamma_m + \dot{\gamma_m} = i_\chi(\bar{t}) \qquad m = 1, \ldots, q \tag{3.8}$$

where $\phi_m \equiv \phi_m(\bar{t})$ is as given by equation (3.4), and i_χ denotes the average current along the χ-direction. The current per staircase in the α-direction can be calculated using the following formula.

$$i_\alpha(\bar{t}) = \frac{1}{q}\sum_{m=1}^{q} \sin\phi_m \cos\gamma_m + \dot{\phi}_m. \tag{3.9}$$

Now we can analyse the dynamics of phase-slips. Since we have voltage driving along the α-direction, we can put $i_\chi(\bar{t}) = 0$. Thus

$$\cos(\phi_m)\sin\gamma_m + \dot{\gamma_m} = 0 \qquad m = 1, \ldots, q. \tag{3.10}$$

The solution $\{\gamma_m = 0\}$ is stable until one of the $\cos(\phi_m)$ becomes negative. Suppose the nth staircase is the first to reach that instability at time $\bar{t} = \bar{t}_0$. However small $\gamma_n(\bar{t}_0)$ may be, due to the instability of equation (3.10) for γ_n, γ_n will grow until $\gamma_n = \pi$, at which point it will regain stability. This is equivalent to ϕ_n changing by π. First we estimate the relaxation time or phase-jumping time using equation (3.10). We set $\bar{t}_0 = 0$ as the time at which $\cos(\phi_n)$ becomes negative; without loss of generality we take $\phi_n = \pi/2$ at $\bar{t}_0 = 0$. We can integrate equation (3.10) to obtain

$$\tan\frac{\gamma_n}{2} = \tan\frac{\gamma_n(0)}{2}\exp\left[-\int_0^{\bar{t}}\cos(\phi_n)\,\mathrm{d}\bar{t}\right] \tag{3.11}$$

with the initial condition $\phi_n(\bar{t} = 0) = \pi/2$. This recalls equation (2.9) for the two-junction system. Following the derivation of equation (2.21), and using equation (3.5) for the driving voltage, we get

$$\cos(\phi_n(\bar{t})) \approx -v_0\bar{t} \tag{3.12}$$

where we assumed the limit of small argument in the sine function and $v_1 \ll v_0$. Now if we put $\bar{\tau}_S$ as the time for a phase-slip to occur, then from

$$\int_0^{\bar{\tau}_S}\cos(\phi_n)\,\mathrm{d}\bar{t} \sim -v_0\bar{\tau}_S^2/2 \tag{3.13}$$

and equation (3.11), we get

$$\bar{\tau}_S \approx \sqrt{\frac{2}{v_0}}\sqrt{\log\left[\tan\frac{\gamma_n(\bar{\tau}_S)}{2}\Big/\tan\frac{\gamma_n(0)}{2}\right]} \tag{3.14}$$

with $\gamma_n(\bar{\tau}_S) \to \pi$ and $\gamma_n(0) \to 0$. In this formula, the logarithmic factor can be considered to be a number of order unity when realistic cutoff values are used for limiting phases $\gamma_n(\bar{\tau}_S)$, and $\gamma_n(0)$. Hence we see that $\bar{\tau}_S$ is mainly determined by its inverse square root dependence on v_0, as in the two-junction system; in fact, equation (3.14) is identical to equation (2.22) for the phase slip in the two-junction system.

Let us suppose that an external DC plus AC voltage is applied at the frequency $\bar{\omega}$. Then the typical DC voltage v_0 at which we might observe Shapiro step structures will be $v_0 \sim \bar{\omega}$ (or voltage per junction $V_j \sim \hbar\omega/2e$ in physical units.) Thus the typical time scales due to both the external AC voltage and the Josephson oscillations are $\sim 1/\bar{\omega}$, while the relaxation time $\bar{\tau}_S \sim \sqrt{1/\bar{\omega}}$. Therefore, we see that the relaxation time can be made much smaller than other time scales if we deal with a small enough driving frequency (compared to ω_R) and hence with small time-averaged voltages. This is the adiabatic limit.

Of course, this computation is not necessarily realistic as a description of the full dynamics of the array. If we enforce spatial periodicity of γ_m through $\gamma_{m+q} = \gamma_m$, then phase-slips occur in the same direction in all parts of the array, which will induce a Hall voltage in the direction transverse to the applied voltage, with its magnitude proportional to the transverse dimension of the array. Many boundary conditions will exclude this possibility.

4. Discussion

The relevance of these results to the more physical case of current-driven arrays is highly debatable. For the two-junction system, the phase-slips that lead to the subharmonic lockings are simply impossible if the system is current-driven. Similarly, for the arrays, the phase-slips would lead to macroscopic Hall voltages across the entire sample. Only if the system can somehow adjust the dynamics (breaking the staircase symmetry!) so as to evade these voltages will the phase-slips occur.

In comparing these results with the general absence of subharmonic lockings in experimental studies of arrays, we are thus driven to one of two conclusions. The first possibility is that the current-driven arrays are able to phase slip as one approaches the adiabatic limit; one would then conclude that present experiments are too far from the adiabatic limit to see the subharmonic lockings. The second, and more interesting possibility, is that the phase-slips are suppressed by the current-driven dynamics in an array, just as they are in a two-junction system. This implies that the dissipative current-driving mechanism successfully maintains the array in a state of higher internal energy than the array would otherwise seek. If true, this would be a remarkable effect.

Acknowledgments

We are grateful to H Jaeger, B Kim and K K Likharev for helpful discussions. This work was supported by the Science and Technology Center for Superconductivity.

Appendix. Josephson mode approximation

In this appendix we will derive an approximate formula for the time-averaged currents for a given external voltage and initial phase configuration of a two-junction system. The amplitude of the zero frequency component of $i(\bar{t})$ from equation (2.8*a*) corresponds to the time-averaged current for a given voltage and initial phase configuration. But it is very difficult to estimate the time-averaged current from the full expression, because it involves a sum of infinite products. This is due to the fact that in section 2 we retained all of the sinusoidal terms in $G(\bar{t})$. If we can approximate $G(\bar{t})$ by only one AC component, then currents may be estimated in a relatively straightforward manner. From the expression for $G(\bar{t})$ for a given AC plus DC voltage driving, (we use the notation of section 2),

$$-2G(\bar{t}) = D + \sum_{m=-\infty}^{\infty} F_m \sin\left[(v_0 - m\bar{\omega})\bar{t} + \beta_0 + \frac{m\pi}{2}\right] \qquad \text{(A1)}$$

where

$$\begin{aligned} D &\equiv 2 \sum_{m=-\infty}^{\infty} \frac{J_m(-v_1/\bar{\omega})}{(v_0 - m\bar{\omega})} \sin\left(\beta_0 + \frac{m\pi}{2}\right) \\ F_m &\equiv -2 \frac{J_m(-v_1/\bar{\omega})}{(v_0 - m\bar{\omega})} \end{aligned} \qquad \text{(A2)}$$

we see that the $m = 0$ term is dominant when $|v_1/\bar{\omega}| \ll 1$ or in the adiabatic limit where $v_0 \ll 1$. This assumed, we can approximate $-2G(\bar{t})$ as

$$-2G(\bar{t}) \approx D + F_0 \sin(v_0\bar{t} + \beta_0) \tag{A3}$$

with D and F_0 as given above. In a sense this can be called a simple Josephson mode approximation or small AC component approximation because of the condition $v_1\bar{\omega} \ll 1$. If we substitute the expression for $-2G(\bar{t})$ into equation (2.8*a*) for $i(\bar{t})$, then we can obtain the time-averaged current by taking the zero frequency component.

If v_0 and $\bar{\omega}$ are rationally related to one another, i.e. if $m\bar{\omega} = nv_0$ for some integral m, n, then we can show that the current depends both upon y_0 (through c_0) and upon the phase-shift with the driving field, β_0. Recall the expression for the average current,

$$\langle i \rangle = \langle \dot{\phi}_2/2 \rangle + \left\langle \sin\left(\frac{\phi_2}{2}\right) \frac{1 - c_0^2 \exp(-2G)}{1 + c_0^2 \exp(-2G)} \right\rangle. \tag{A4}$$

Here $\langle\,\rangle$ denotes a time average. In the second term, $\sin(\phi_2/2)$ can be expressed as

$$\begin{aligned} \sin\left(\frac{\phi_2}{2}\right) &= \sin\left[\beta_0 + v_0\bar{t} - \frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t})\right] \\ &= \mathrm{Im}\left[\exp(\mathrm{i}\beta_0)\exp(\mathrm{i}v_0\bar{t})\exp\left(-\mathrm{i}\frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t})\right)\right] \end{aligned} \tag{A5}$$

which can be written as

$$\sin\left(\frac{\phi_2}{2}\right) = \mathrm{Im}\left[\exp(\mathrm{i}\beta_0)\exp(\mathrm{i}v_0\bar{t}) \sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m\left(-\frac{v_1}{\bar{\omega}}\right)\exp(\mathrm{i}m\bar{\omega}\bar{t})\right]. \tag{A6}$$

The second term in the supercurrent part of equation (A4), the factor depending on $G(\bar{t})$, can also be expanded in a Bessel series. In general, we can write

$$\frac{1 - c_0^2 \exp(-2G)}{1 + c_0^2 \exp(-2G)} = \sum_{n=-\infty}^{\infty} G_n \exp[\mathrm{i}n(v_0\bar{t} + \beta_0)]. \tag{A7}$$

If $m\bar{\omega} = -nv_0$, then there will be a contribution $i_{m,-n}$ to the average current from the term

$$i_{m,-n} = \mathrm{Im}\left[\exp(\mathrm{i}\beta_0)\mathrm{i}^m J_m\left(\frac{v_1}{\bar{\omega}}\right) G_{-n}\right] \tag{A8}$$

which explicitly depends upon both y_0 (through G_{-n}) and β_0.

If v_0 and $\bar{\omega}$ are irrationally related to each other, the expression for $i(\bar{t})$ within this approximation has a nice property. Let us suppose that is the case. Note that in equation (A4) the factor $\exp[-\mathrm{i}(v_1/\bar{\omega})\cos(\bar{\omega}\bar{t})]$ is a periodic function with frequency $\bar{\omega}$, and the remaining factor on the right-hand side is a periodic function with frequency v_0, due to our approximation. Therefore, if we assume that $\bar{\omega}$ and v_0 are irrationally related, then the only way to get a zero frequency component is to

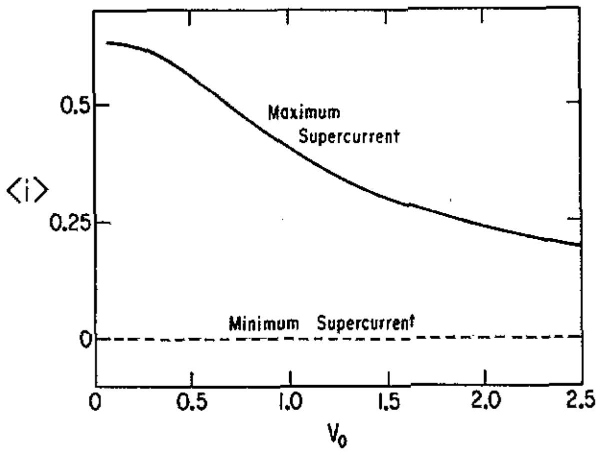

Figure 4. Maximum and minimum time-averaged supercurrent against DC voltage per junction in a two-junction system under DC voltage driving, with $v_1 = 0$. The current is measured in units of I_c, and the voltage in units of $I_c R$.

multiply the zero frequency part of each of the two factors. Taking the zero frequency mode of $\exp[-\mathrm{i}(v_1/\bar{\omega})\cos(\bar{\omega}\bar{t})]$

$$\left\langle \exp\left[-\mathrm{i}\frac{v_1}{\bar{\omega}}\cos(\bar{\omega}\bar{t})\right]\right\rangle = J_0\left(\frac{-v_1}{\bar{\omega}}\right) \tag{A9}$$

and

$$\left\langle \sin(\mathrm{i}(v_0+\beta_0)\bar{t})\frac{1-c_0^2\exp(-2G)}{1+c_0^2\exp(-2G)}\right\rangle = \frac{1}{\bar{\tau}}\int_0^{\bar{\tau}} \mathrm{d}\bar{t}\,\sin(\mathrm{i}(v_0+\beta_0)\bar{t})\frac{1-c_0^2\exp(-2G)}{1+c_0^2\exp(-2G)} \tag{A10}$$

with $-2G(\bar{t})$ as given in equation (A3), and $\bar{\tau} = 2\pi/v_0$ the period of the integrand. Setting $d = c_0^2\exp(D)$, and $\phi = (v_0\bar{t} + \beta_0)$, some simple manipulations lead to

$$\frac{1}{\bar{\tau}}\int_0^{\bar{\tau}} \mathrm{d}t\,\sin\phi\frac{1-c_0^2\exp(-2G(\phi))}{1+c_0^2\exp(-2G(\phi))} = \frac{1}{v_0}\frac{d}{dF_0}\int_0^{\pi}\mathrm{d}\phi\,\log\left(1+d^2+2d\cosh(F_0\sin\phi)\right) \tag{A11}$$

which explicitly does not depend upon the phase β_0, although it does depend on y_0 through d. Actually, there is a dependence of d on β_0, but this is only a consequence of the fact that changing β_0 also changes the defined zero of time, and thus the value of y_0. This can be most easily seen by absorbing β_0 into the phase of the AC driving. This can be used for approximate numerical evaluation of $\langle i\rangle$ (see figure 4). Actually, as shown in figure 4, the current depends upon y_0 even in the case where the AC driving $v_1 = 0$.

For irrationally related $\bar{\omega}$ and v_0, it is possible from the above to prove one feature of the behaviour of $\langle i\rangle$ if $v_1 = 0$. It is easy to show from equation (A11) that, when $v_0 > 0$, $\langle i \rangle - v_0 > 0$ when $v_0 < 0$, $\langle i \rangle - v_0 < 0$. Thus the net supercurrent at irrational values of the voltage in this case is always of the same sign as the normal current (see figure 4).

References

[1] Lindelof P E 1981 *Rep. Prog. Phys.* **44** 949
Imry Y 1983 *Statics and Dynamics of Nonlinear Systems* ed G Benedek, H Bilz and R Zeyher (Berlin: Springer) p 170

[2] Bohr T, Bak P and Jensen M H 1984 *Phys. Rev.* A **30** 1970, and references therein

[3] Belykh V N, Pedersen N F and Soerensen O H 1978 *Phys. Rev.* B **16** 4860

[4] Tinkham M, Abraham D W and Lobb C J 1983 *Phys. Rev.* B **28** 6578
Webb R A, Voss R F, Grinstein G and Horn P M 1983 *Phys. Rev. Lett.* **51** 690

[5] Teitel S and Jayaprakash C 1983 *Phys. Rev. Lett.* **51** 1999
Shih W and Stroud D 1983 *Phys. Rev.* B **28** 6575

[6] Teitel S and Jayaprakash C 1983 *Phys. Rev.* B **27** 598
Halsey T C 1985 *J. Phys. C: Solid State Phys.* **18** 2437

[7] Leeman C, Lerch P and Martinoli P 1984 *Physica* **126B** 475
Clark T D 1973 *Phys. Rev.* B **8** 137

[8] Benz S P, Rzchowski M S, Tinkham M and Lobb C J 1990 *Phys. Rev. Lett.* **64** 693

[9] Lee H C, Mast D B, Newrock R S, Bortner L, Brown K, Esposito F P, Harris D C and Garland J C 1990 *Physica* **165–6B** 1571

[10] Lee K H, Stroud D and Chung J S 1990 *Phys. Rev. Lett.* **64** 962
Lee K H and Stroud D 1991 *Phys. Rev.* **43** 5280

[11] Halsey T C 1990 *Phys. Rev.* B **41** 11 634

[12] Halsey T C 1985 *Phys. Rev.* B **31** 5728; 1988 *Physica* B **152** 22

[13] Lee H C, Newrock R S, Mast D B, Hebboul S E, Garland J C and Lobb C J 1991 *Phys. Rev.* B **44** 921

[14] Sohn L L, Rzchowski M S, Free J U, Benz S P, Tinkham M and Lobb C J 1991 *Phys. Rev.* B **44** 925

[15] Kvale M and Hebboul S E 1991 *Phys. Rev.* B **43** 3720

[16] Josephson B D 1962 *Phys. Lett.* **1** 251